A Graph on Bumpless Pipe Dreams

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Featuring joint work with:

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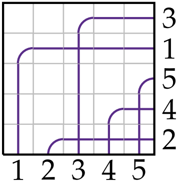
Bumpless Pipe Dreams

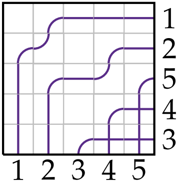
Definition (Lam–Lee–Shimozono 2018)

A *bumpless pipe dream* is a filling of an $n \times n$ grid using the tiles

 and 

such that no two pipes cross more than once.

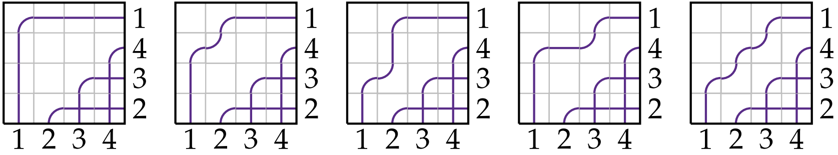




Order pipes exit from top-to-bottom is a *permutation*, written w.

BPD(1432)

Fix exit order to be $w = 1432$:



How do we know that we've found *all* of them?

BPD(1432)

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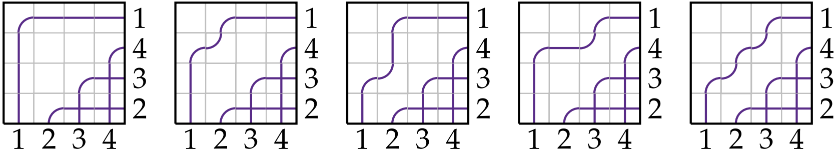

How do we know that we've found *all* of them?

Theorem (Lam–Lee–Shimozono 2018)

For any w, the set BPD(w) *can be generated via "droop moves".*

Droop moves

Definition

A *droop move*  ↘  is valid if

there are no other  or  tiles in between.

Re-routing a pipe — same exit order

Droop moves

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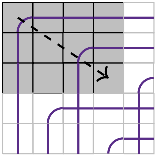
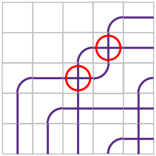
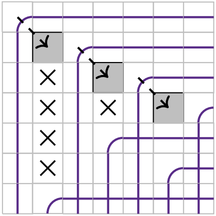




not valid — the  in between ⇝ double-crossing

Minimal Droops

We call a droop *minimal* if it is “as close as possible”



Any droop move is a sequence of min-droop moves

BPD Graph

Definition (Gregory, P. 2025+)

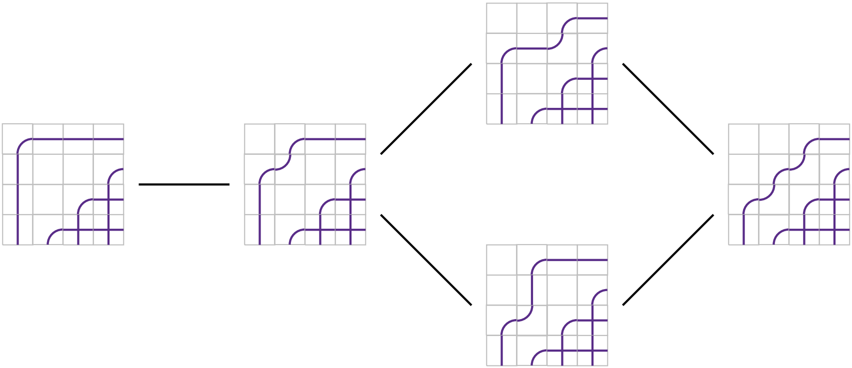
The *bumpless pipe dream graph* of a permutation w is $H_w = (V_w, E_w)$ where $V_w = \mathrm{BPD}(w)$ and

$$\{B_1, B_2\} \in E_w \iff B_1 \text{ and } B_2 \text{ related by } \textit{min-droop}.$$

Corollary (Lam–Lee–Shimozono 2018)

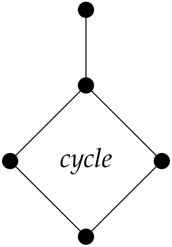
The graph H_w is connected for any w.

BPD Graph

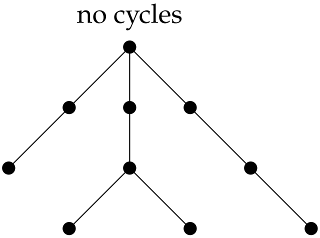


The bumpless pipe dream graph H_w for $w = 1432$

Cycles in BPD Graphs



BPD graph for 1432



BPD *tree* for 321645

Research Question

Determine whether H_w will have a cycle **just by looking at** w.

Main Result

Theorem (Gregory, P. 2025+)

For the BPD graph H_w to contain a cycle, it is sufficient for w to contain one of the following fifteen patterns:

$$1432, 12453, 12534, 13254, 13524, 14253, 14523, 24153, 31524,$$

$$35142, 214563, 216345, 354162, 461325, \textit{ and } 456132.$$

Example (Our result is independent of n)

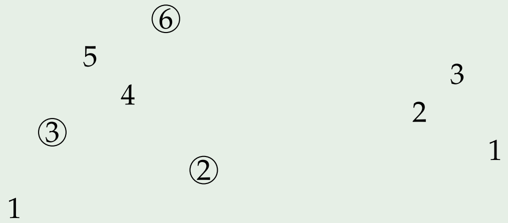
If we have some permutation of 81 numbers, we need only check if it contains one of our fifteen patterns.

Pattern Avoidance

Example

The permutation $w = 1\,\overline{3}\,5\,4\,\overline{6}\,\overline{2}$

- *contains* u = 231 as a pattern (e.g., the 342);
- but *avoids* v = 312.



Conclusion

Research Methods

- implemented objects in SageMath (`python`-based)
- analyzed data for $\sum_{k=1}^{9} k! = 409{,}113$ graphs to form conjecture

Completed Work

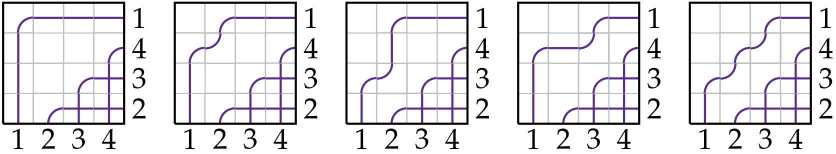
- proved it is necessary to avoid these patterns

Future Directions

- show it is also sufficient to avoid these patterns
- investigate other properties of these graphs

Schubert polynomials

Fix exit order to be $w = 1432$:



Computes the *Schubert polynomial*

$$\mathfrak{S}_{1432}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_2^2 x_3.$$

- applications to geometry and theoretical physics